Test 3

Calculus of Trigonometric Functions
Discrete Random Variables
Binomial Distributions

Semester One 2018

Year 12 Mathematics Methods

Calculator Assumed

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You may have a formula sheet for this section of the test.
Classpad Calculators
1 page of Notes

Total________/41 45 minutes +5 minutes READING

Question 1 **(5 marks)**

The discrete random variable X has the probability distribution shown in the table below.

x	0	1	2	3
$P(X=x)$	$\frac{2a^2}{3}$	$\frac{1-3a}{3}$	$\frac{1+2a}{3}$	$\frac{4a^2}{3}$

Determine the value of the constant a.

$$\frac{2a^2}{3}+\frac{1-3a}{3}+\frac{1+2a}{3}+\frac{4a^2}{3}=1$$

$$6a^2-a-1=0$$

$$(2a-1)(3a+1)=0$$

$$\therefore a=\frac{1}{2} \text{ or } a=-\frac{1}{3}$$

Check: $a=\frac{1}{2}$, $\frac{1-3a}{3}<0$, reject $a=\frac{1}{2}$

$$\therefore a=-\frac{1}{3}$$

If $a=\frac{1}{2}$, or $a=-\frac{1}{3}$ only

-2

Question 2 **(8 marks)**

(a) Differentiate $e^{-3x}\sin(2x)$ with respect to x, showing full working. (2 marks)

$$\frac{d}{dx} e^{-3x}\sin(2x)$$

$$= -3e^{-3x}\sin 2x + e^{-3x}\cos(2x) \times 2$$

$$= e^{-3x}(-3\sin 2x + 2\cos 2x)$$

(b) Hence find the following indefinite integral. (3 marks)

$$-3\int e^{-3x}\sin(2x)\,dx + 2\int e^{-3x}\cos(2x)\,dx.$$

And using a similar process as part (a), find the indefinite integral for

$$-3\int e^{-3x}\cos(2x)\,dx - 2\int e^{-3x}\sin(2x)\,dx.$$

By (a), $\int -3e^{-3x}\sin 2x + 2e^{-3x}\cos(2x)\,dx$

$$= e^{-3x}\sin(2x) + C_1$$

$$\frac{d}{dx} e^{-3x}\cos(2x) = -3e^{-3x}\cos(2x) - e^{-3x}\sin(2x)\times 2$$

$$\therefore \int -3e^{-3x}\cos(2x)\,dx - \int 2e^{-3x}\sin(2x)\,dx$$

$$= e^{-3x}\cos(2x) + C_2$$

If Answer $\left\{ \dfrac{3(2\cos(2x) + 3\sin(2x))e^{-3x}}{13} - \dfrac{2(3\cos(2x) - 2\sin(2x))e^{-3x}}{13} \right.$

* Should use Interactive Simplify answer

$\Longrightarrow \sin(2x)\,e^{-3x} + c$

(c) Use the two equations from (b) to determine $\int e^{-3x} \sin(2x)\, dx$. (3 marks)

$$e^{-3x}\sin(2x) + C_1 = -3\int e^{-3x}\sin(2x)\,dx + 2\int e^{-3x}\cos(2x)\,dx \quad (1)$$

$$e^{-3x}\cos(2x) + C_2 = -3\int e^{-3x}\cos(2x)\,dx - 2\int e^{-3x}\sin(2x)\,dx \quad (2)$$

$(1) \times 3$: $3e^{-3x}\sin(2x) + 3C_1$

$$= -9\int e^{-3x}\sin(2x)\,dx + 6\int e^{-3x}\cos(2x)\,dx \quad (3)$$

$(2) \times 2$: $2e^{-3x}\cos(2x) + 2C_2$

$$= -6\int e^{-3x}\cos(2x)\,dx - 4\int e^{-3x}\sin(2x)\,dx \quad (4)$$

$(3) + (4)$: $3e^{-3x}\sin(2x) + 2e^{-3x}\cos(2x) + C$

$$= -13\int e^{-3x}\sin(2x)\,dx \quad ✓✓$$

$$\therefore \int e^{-3x}\sin(2x)\,dx = -\frac{1}{13}\left(3e^{-3x}\sin(2x) + 2e^{-3x}\cos(2x) + \right.$$

✓

* If they don't "Use the two equations from (b)"

1 mark Only

Question 3 **(6 marks)**

Differentiate with respect to x, (show full working)

(a) $y = \sin^3(2x+1)$. (3 marks)

$$\frac{dy}{dx} = 3\sin^2(2x+1)\cos(2x+1) \times 2$$

$$= 6\sin^2(2x+1)\cos(2x+1)$$

Evaluate the following, showing full working.

(b) $\int_{\pi/6}^{\pi/2} \cos(2x)\,dx$ (3 marks)

$$= \frac{1}{2}\sin(2x)\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$ (Find anti-derivative)

$$= \frac{1}{2}\left(\sin\pi - \sin\frac{\pi}{3}\right)$$ (Evaluate by substitution)

$$= \frac{1}{2}\left(0 - \frac{\sqrt{3}}{2}\right)$$

$$= -\frac{\sqrt{3}}{4}$$ (Find answer)

$$\doteq -0.4330$$

Question 4

(9 marks)

75% of the avocados produced by a farm are known to be first grade, the rest being second grade. Trays of 24 avocados are filled at random in a packing shed and sent to market.

Let the random variable X be the number of first grade avocados in a single tray.

(a) Explain why X is a discrete random variable, and identify its probability distribution.

(2 marks)

X is a DRV as it can only take integer values from 0 to 24. ✓

$X \sim B(24, 0.75)$ which is a binomial distribution ✓

(b) Calculate the mean and standard deviation of X.

(2 marks)

$\bar{X} = 24 \times 0.75 = 18$ ✓

$\sigma_x = \sqrt{18 \times 0.25} = \frac{3\sqrt{2}}{2} \approx 2.12$ ✓

(c) Determine the probability that a randomly chosen tray contains

(i) 18 first grade avocados.

(1 mark)

$P(X = 18) = 0.1853$ ✓

(ii) more than 15 but less than 20 first grade avocados.

(2 marks)

$P(15 < X < 20)$

$= P(16 \leq X \leq 19)$ ✓ $= 0.6320$ ✓

(d) In a random sample of 1000 trays, how many trays are likely to have fewer first grade than second grade avocados.

(2 marks)

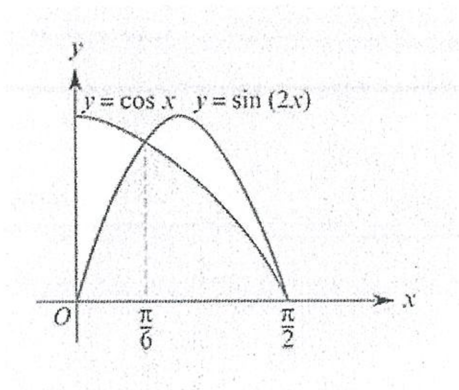
$P(X \leq 11) = 0.0021$ ✓

If trays of 24 then X must be <12 ie $P(X \leq 11)$

$0.0021 \times 1000 = 2.1 \approx 2$ trays. ✓

Question 5 **(4 marks)**

Find the area between the two curves from $0 \leq x \leq \frac{\pi}{2}$, showing full algebraic reasoning.



$$\int_0^{\frac{\pi}{6}} \left| \cos x - \sin 2x \right| dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left| \sin 2x - \cos x \right| dx$$

$$= \left[\sin x + \frac{\cos(2x)}{2} \right]_0^{\frac{\pi}{6}} + \left[-\frac{\cos 2x}{2} - \sin x \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left[\left(\frac{1}{2} + \frac{1}{4} \right) - \left(0 + \frac{1}{2} \right) \right] + \left[\left(\frac{1}{2} - 1 \right) - \left(-\frac{1}{4} - \frac{1}{2} \right) \right]$$

$$= \frac{1}{4} + \frac{1}{4}$$

$$= \frac{1}{2}$$

Question 6 **(9 marks)**

(a) A sample of six objects is to be drawn from a large population in which 20% of the objects are defective. Find the probability that the sample contains:

(i) three defectives. (2 marks)

$X \sim Bin(6, 0.2)$

$P(X=3) = \text{binomial PDF}(3, 6, 0.2) = 0.08192$ ✓✓

(ii) fewer than three defectives. (2 marks)

$X \sim Bin(6, 0.2)$

$P(X \leq 2) = \text{binomial CDF}(0, 2, 6, 0.2) = 0.90112$ ✓✓

(b) Another large population contains a proportion p of defective items.

(i) Write down an expression in terms of p for P, the probability that a sample of six items contains exactly two defectives. (2 marks)

$P = {}^6C_2\, p^2(1-p)^4 = \underline{\underline{15p^2(1-p)^4}}$ ✓✓

(ii) By differentiating to find $\frac{dP}{dp}$, show that P is greatest when $p = \frac{1}{3}$. (3 marks)

$$\frac{dP}{dp} = 15 \times 2p \times (1-p)^4 + 15p^2 \times 4(1-p)^3 \times (-1)$$ ✓

$$= 30p(1-p)^4 - 60p^2(1-p)^3$$

$$= 30p(1-p)^3((1-p) - 2p)$$

$$= 30p(1-p)^3(1-3p) = 0$$

$p = 0, 1, \frac{1}{3}$ ✓

reject $p = 0, 1$ as p is proportion

$\therefore p = \frac{1}{3}$ ✓

$\frac{d^2P}{dp^2}(0) = 30$ +ve $\therefore$ min

$\frac{d^2P}{dp^2}(1) = 0$ $\therefore$ P.O.I.

$\frac{d^2P}{dp^2}\left(\frac{1}{3}\right) = \frac{-80}{9}$ −ve $\therefore$ MAX

$90p^5 - 300p^4 + 360p^3 - 180p^2 + 30p$